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Center conditions: rigidity of logarithmic differential equations[☆]

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Dedicated to Mothers

Abstract

In this paper, we prove that any degree d deformation of a generic logarithmic polynomial differential equation with a persistent center must be logarithmic again. This is a generalization of Ilyashenko's result on Hamiltonian differential equations. The main tools are Picard–Lefschetz theory of a polynomial with complex coefficients in two variables, specially the Gusein-Zade/A'Campo's theorem on calculating the Dynkin diagram of the polynomial, and the action of Gauss–Manin connection on the so-called Brieskorn lattice/Petrov module of the polynomial. We will also generalize J.P. Francoise recursion formula and (*) condition for a polynomial which is a product of lines in a general position. Some applications on the cyclicity of cycles and the Bautin ideals will be given.

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0. Introduction

Let us be given the 1-form

$$\omega = P(x, y) dy - Q(x, y) dx, \quad (1)$$

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where P and Q are two relatively prime polynomials in \mathbb{C}^2 . The degree of ω is the maximum of $\deg(P)$ and $\deg(Q)$. The space of degree d ω 's up to multiplication by a constant, namely $\mathcal{F}(d)$, is a Zariski open subset of the projective space associated to the coefficient space of polynomial 1-forms (1) with $\deg(P), \deg(Q) \leq d$. An element of $\mathcal{F}(d)$ induces a holomorphic foliation \mathcal{F} on \mathbb{C}^2 , i.e., the restrictions of ω to the leaves of \mathcal{F} are identically zero. Therefore, we denote an element of $\mathcal{F}(d)$ by $\mathcal{F}(\omega)$ or \mathcal{F} if there is no confusion about the underlying 1-form in the text and we say that it is of degree d .

The points in $\text{Sing}(\mathcal{F}(\omega)) = \{P = 0, Q = 0\}$ are called the singularities of \mathcal{F} . A singularity $p \in \mathbb{C}^2$ of $\mathcal{F}(\omega)$ is called reduced if $(P_x Q_y - P_y Q_x)(p) \neq 0$. A reduced singularity p is called a center singularity or center for simplicity if there is a holomorphic coordinates system (\tilde{x}, \tilde{y}) around p with $\tilde{x}(p) = 0, \tilde{y}(p) = 0$ and such that in this coordinates system $\omega \wedge d(\tilde{x}^2 + \tilde{y}^2) = 0$. Let $\mathcal{M}(d)$ be the closure of the subset of $\mathcal{F}(d)$ containing $\mathcal{F}(\omega)$'s with at least one center. It is a well-known fact that $\mathcal{M}(d)$ is an algebraic subset of $\mathcal{F}(d)$ (see for instance [Mo1]). Now the problem of identifying irreducible components of $\mathcal{M}(d)$ arises. This problem is also known by the name “Center conditions” in the context of real polynomial differential equations.

Dulac in [Du] proves that $\mathcal{M}(2)$ has exactly nine irreducible components (see also [CL, p. 601]). In this case any foliation in $\mathcal{M}(2)$ has a Liouvillian first integral. Since this problem finds applications on the number of limit cycles in the context of real differential equations, this classification problem is very active. It is recommended to the reader to do a search with the title center/center conditions in mathematical review to obtain many recent papers on this problem. We find some partial results for $d = 3$ due to H. Zoladek and others and a similar problem for Abel equations $y' = p(x)y^2 + q(x)y^3$, p, q polynomials in x (see also [BFY] and its references). In the context of holomorphic foliations we can refer to [CL, Muc, Mo1]. One of the main objectives in this paper is to introduce some new methods in this problem using an elementary algebraic geometry. We have borrowed many notions like Brieskorn modules, Picard–Lefschetz theory and so on from the literature of singularities of holomorphic functions (see [AGV]).

Let us be given the polynomials $f_i, \deg(f_i) = d_i, 1 \leq i \leq s$ and non-zero complex numbers $\lambda_i \in \mathbb{C}^*, 1 \leq i \leq s$. The foliation

$$\mathcal{F} = \mathcal{F} \left(f_1 \cdots f_s \sum_{i=1}^s \lambda_i \frac{df_i}{f_i} \right) \quad (2)$$

is of degree $d = \sum_{i=1}^s d_i - 1$ and has the logarithmic first integral $f_1^{\lambda_1} \cdots f_s^{\lambda_s}$. Let $\mathcal{L}(d_1, \dots, d_s)$ be the set of foliations (2). Since $\mathcal{L}(d_1, \dots, d_s)$ is parameterized by λ_i 's and the coefficients of f_i 's, it is irreducible. The main theorem of this paper is:

Theorem 0.1. $\overline{\mathcal{L}(d_1, \dots, d_s)}$ is an irreducible component of $\mathcal{M}(d)$, where $d = \sum_{i=1}^s d_i - 1$ and $\overline{\mathcal{L}(d_1, \dots, d_s)}$ is the closure of $\mathcal{L}(d_1, \dots, d_s)$ in $\mathcal{F}(d)$.

In the case $s = 1$ we can assume that $\lambda_1 = 1$ and so $\mathcal{L}(d+1)$ is the space of foliations of the type $\mathcal{F}(df)$, where f is a polynomial of degree $d+1$. This case is proved by Ilyashenko in [II]. The similar result for foliations with a first integral of the type $\frac{Fp}{Gq}, \frac{\deg(F)}{\deg(G)} = \frac{q}{p}, g.c.d.(p, q) = 1$ is obtained in [Mo, Mo1]. Some basic tools of this kind of generalizations for Lefschetz pencils on a manifold is worked in [Muc].

Let us reformulate our main theorem as follows: Let $\mathcal{F} \in \mathcal{L}(d_1, \dots, d_s)$ be given by (2), p one of the center singularities of \mathcal{F} and \mathcal{F}_ε a holomorphic deformation of \mathcal{F} in $\mathcal{F}(d)$ such that its unique singularity p_ε near p is still a center. There exists an open dense subset U of $\mathcal{L}(d_1, \dots, d_s)$, such that for all $\mathcal{F} \in U$, \mathcal{F}_ε admits a logarithmic first integral. More precisely, there exist polynomials $f_{i\varepsilon}, \deg(f_{i\varepsilon}) = d_i, i = 1, \dots, s$ and non-zero complex numbers $\lambda_{i\varepsilon}$ such that \mathcal{F}_ε is given by

$$f_{1\varepsilon} \cdots f_{s\varepsilon} \sum_{i=1}^s \lambda_{i\varepsilon} \frac{df_{i\varepsilon}}{f_{i\varepsilon}} = 0$$

where $f_{i\varepsilon}$ and $\lambda_{i\varepsilon}$ are holomorphic in ε and $f_{i0} = f_i, \lambda_{i0} = \lambda_i, i = 1, \dots, s$. This new formulation of our main theorem says that the persistence of one center implies the persistence of all other centers and dicritical singularities $\{f_i = 0\} \cap \{f_j = 0\}, i, j = 1, \dots, s$.

We can put U the complement of $\mathcal{L}(d_1, \dots, d_s) \cap \text{sing}(\mathcal{M}(d))$ in $\mathcal{L}(d_1, \dots, d_s)$. One may not be satisfied with this U and try to find explicit conditions, for instance: A foliation $\mathcal{F}(f_1 \cdots f_s \sum_{i=1}^s \lambda_i \frac{df_i}{f_i})$ in U satisfies 1. $\{f_i = 0\}$ intersects $\{f_j = 0\}$ transversally 2. $f_1^{\lambda_1} \cdots f_s^{\lambda_s}$ has non-degenerated critical points in $\mathbb{C}^2 - \bigcup_{i=1}^s \{f_i = 0\}$ and so on. In general, one may be interested to identify the set $\mathcal{L}(d_1, \dots, d_s) \cap \text{sing}(\mathcal{M}(d))$. In any case these questions are not in the focus of this paper.

Since this paper is inspired by Ilyashenko's paper [II] let us give a sketch of the proof in the case $\mathcal{L}(d+1)$: Let f be a degree $d+1$ polynomial with the following conditions: (1) f has d^2 non-degenerate critical points with distinct values, (2) the homogeneous part of f of degree $d+1$ has $d+1$ distinct roots. These conditions are generic, i.e. in the space of polynomials of degree $d+1$ there is an open dense subset such that for all f in this subset the conditions are satisfied. Let $df + \varepsilon\omega + h.o.t.$ be a deformation of df such that the singularity near a center singularity of df , namely p_1 , persists in being center. Then $\int_\delta \omega = 0$ for all vanishing cycles δ around p_1 . The action of the monodromy on a single vanishing cycle $\delta \in H_1(f^{-1}(b), \mathbb{Z})$, where b is a regular value of f , generates the whole homology (the most significant part of the proof). Therefore, our integral is zero in all cycles in the fibers of f and so it is relatively exact. Knowing the fact $\deg(\omega) \leq d$ we conclude that $\omega = dP$, where P is a polynomial of degree less than $d+1$. Since $\mathcal{M}(d)$ is an algebraic set, the hypothesis on df is generic and the tangent vector ω of any deformation of df in $\mathcal{M}(d)$ is tangent also to $\mathcal{L}(d+1)$, we conclude that $\mathcal{L}(d+1)$ is an irreducible component of $\mathcal{F}(d)$.

Now let us explain our strategy for the proof of Theorem 0.1 and the structure of the paper. First of all, since Picard–Lefschetz theory and classification of relatively exact 1-forms of a multi-valued function $f_1^{\lambda_1} \dots f_s^{\lambda_s}$ are not well developed, it seems to be difficult to take a generic element of $\mathcal{L}(d_1, \dots, d_s)$ and then try to repeat Ilyashenko’s argument. So we look for a special point in $\mathcal{L}(d_1, \dots, d_s)$. This special point is going to be $\mathcal{F}_0 = \mathcal{F}_0(df)$, $f = \prod_{i=1}^d l_i$, where l_i is a polynomial of degree one in \mathbb{R}^2 and the lines $l_i = 0$ are in general positions in \mathbb{R}^2 . Every $\mathcal{L}(d_1, \dots, d_s)$, $\sum_{i=1}^s d_i = d + 1$ passes through \mathcal{F}_0 and around \mathcal{F}_0 may have many irreducible components. The main point is to prove that the tangent cone of $\mathcal{M}(d)$ in \mathcal{F}_0 is equal to the tangent cone of $\bigcup_{\sum_{i=1}^s d_i = d+1, d_i \in \mathbb{N} \cup \{0\}} \mathcal{L}(d_1, \dots, d_s)$. This will be enough to prove our main theorem. To start these calculations, three important tools are needed which I have put them in Sections 1–3. Roughly speaking, in Section 1 we want to classify rational 1-forms in \mathbb{C}^2 whose derivatives (Gauss–Manin connection) after some certain order is relatively exact. We introduce the Brieskorn lattice/Petrov module H associated to a polynomial f in \mathbb{C}^2 and the action of Gauss–Manin connection ∇ on it. Using a theorem of Mattei–Cerveau we prove Corollary 1.1 which is enough for our needs in this paper. This corollary classifies all $\omega \in H$ with $\nabla^n \omega = 0$ for a given natural number n . In Section 2 we analyze the action of the monodromy on a Lefschetz vanishing cycle in f . Using the well-known Theorem 2.2 and Gusein-Zade/A’Campo’s Theorem 2.1 we prove Theorem 2.3 and then Theorem 2.4. In Section 3 we consider the deformation $df + \varepsilon^k \omega_k + \varepsilon^{k+1} \omega_{k+1} + \dots + \varepsilon^{2k} \omega_{2k} + h.o.t. = 0$, $\omega_k \neq 0, k \in \mathbb{N}$ of $df = 0$ with a persistent center. We calculate the Melnikov functions M_i , $i = 1, \dots, 2k$ and knowing that they are identically zero we will obtain explicit forms of ω_i , $i = k, \dots, 2k$. Section 4 is devoted to the proof of Theorem 0.1. At the end of this section we discuss our result in the context of real differential equations and its connection with concepts like cyclicity and Bautin Ideals.

1. Brieskorn lattices/Petrov modules

Let f be a non-composite polynomial of degree $d + 1$ in \mathbb{C}^2 , i.e. f cannot be composed as $p \circ g$, where p is a polynomial of degree greater than one in \mathbb{C} and g is a polynomial in \mathbb{C}^2 . This condition is equivalent to the fact that for all $b \in \mathbb{C}$ except a finite number the fiber $f^{-1}(b)$ is irreducible (see [Go]). Let $\Omega^i, i = 0, 1, 2$ be the set of polynomial differential i -forms in \mathbb{C}^2 and $\mathbb{C}[t]$ be the ring of polynomials in t . Ω^i is a $\mathbb{C}[t]$ -module in the following way:

$$p(t) \cdot \omega = p(f) \omega, \quad p \in \mathbb{C}[t], \quad \omega \in \Omega^i.$$

The Brieskorn lattice/Petrov module

$$H = \frac{\Omega^1}{d\Omega^0 + \Omega^0 df}$$

is a $\mathbb{C}[t]$ -module. Also, we define

$$V = \frac{\Omega^2}{df \wedge \Omega^1} \cong \frac{\mathbb{C}[x, y]}{\langle f_x, f_y \rangle}.$$

Multiplying by f defines a linear operator on V which is denoted by A .

Lemma 1.1. *If f has isolated singularities then the following are true:*

1. V is a \mathbb{C} -vector space of dimension μ , where μ is the sum of local Milnor numbers of f .
2. Eigenvalues of A are exactly the critical values of f .

Proof. Consider the restriction map $R: V \rightarrow \bigoplus_p \frac{\mathcal{O}_{\mathbb{C}^2, p}}{\langle f_x, f_y \rangle}$, where $\mathcal{O}_{\mathbb{C}^2, p}$ is the ring of germs of holomorphic functions in a neighborhood of p in \mathbb{C}^2 and the sum runs through all critical points of f . By Noether's theorem (see [GrHa, p. 703]), R is an isomorphism (in fact, for surjectivity of R we must modify the proof of Noether's theorem). Each $\frac{\mathcal{O}_{\mathbb{C}^2, p}}{\langle f_x, f_y \rangle}$ is invariant by the linear operator A and $A - f(p) \cdot I$ restricted to it is nilpotent (see [Br]). \square

From now on we assume that f has isolated singularities and we denote the corresponding critical values by c_1, c_2, \dots, c_r . Let \tilde{H} be the localization of H by polynomials in t which vanish only on c_i 's and let $p(t)$ be the minimal polynomial of A . An element of \tilde{H} is a fraction $\omega/a(t)$, $\text{zero}(a(t)) \subset \{c_1, c_2, \dots, c_r\}$ and we have the usual equality $\omega/a(t) = \tilde{\omega}/\tilde{a}(t)$ if $\tilde{a}(t)\omega = a(t)\tilde{\omega}$, between two fractions. The Gauss–Manin connection

$$\nabla: H \rightarrow \tilde{H}$$

is defined as follows: For an $\omega \in H$ we have $p(f)d\omega = 0$ in V . Therefore, there is a polynomial 1-form η in \mathbb{C}^2 such that

$$p(f)d\omega = df \wedge \eta, \quad (3)$$

we define $\nabla\omega = \eta/p(t)$. Of course, we must check that this operator is well defined. If η_1 and η_2 are two polynomial 1-forms satisfying (3) then $(\eta_1 - \eta_2) \wedge df = 0$ and so by de Rham lemma $\eta_1 - \eta_2 = Pdf$ ($= 0$ in H), for a P polynomial in \mathbb{C}^2 . Also if $\omega = dP + Qdf$, P and Q two polynomials in \mathbb{C}^2 , then $d\omega = dQ \wedge df$ and so $\nabla\omega = dQ$ ($= 0$ in H).

We can extend ∇ as a function from \tilde{H} to \tilde{H} by the rule

$$\nabla(\omega/q) = (q\nabla\omega - q'\omega)/q^2. \quad (4)$$

Let $f_1 = 0, f_2 = 0, \dots, f_k = 0$ be irreducible components of all critical fibers of f and $\tilde{\Omega}^i$ be the set of rational i -forms in \mathbb{C}^2 with poles of arbitrary order along $\{f_i = 0\}$'s. We define $\tilde{H}' = \frac{\tilde{\Omega}^1}{d\tilde{\Omega}^0 + \tilde{\Omega}^0 df}$ and in a similar way as for H a connection $\nabla' : \tilde{H}' \rightarrow \tilde{H}'$ given by rule (3).

Lemma 1.2. (\tilde{H}, ∇) is isomorphic to (\tilde{H}', ∇') .

Proof. Every rational 1-form in \mathbb{C}^2 with poles of arbitrary order along $\{f_i = 0\}$'s determines a unique element of \tilde{H} as follows: if $\omega = \frac{\tilde{\omega}}{f_1^{r_1}}$ and $f - c_1 = f_{i_1}^{k_1} \dots f_{i_l}^{k_l}$ is the decomposition of $f - c_1$ to irreducible factors then we multiply both $\tilde{\omega}$ and $f_{i_1}^{r_1}$ by $f_{i_1}^{k_1 m - r_1} f_2^{k_2 m} \dots f_{i_l}^{k_l m}$, where m is an integer number satisfying $m - 1 < \frac{r_1}{k_1} \leq m$, and we obtain $\omega = \frac{\tilde{\omega}}{(f - c_1)^m}$. Repeating this process by $\tilde{\omega}$ we obtain an element of \tilde{H} . If $\omega_1 = dP + Qdf$, where P and Q are two rational functions on \mathbb{C}^2 with poles of arbitrary order along $\{f_i = 0\}$'s then by applying the above method on P, Q we can see that ω is associated to zero in \tilde{H} . Therefore, we obtain a map $\tilde{H}' \rightarrow \tilde{H}$ which is the inverse of the canonical map $\tilde{H} \rightarrow \tilde{H}'$ and so it is an isomorphism. Since ∇ and ∇' coincide on $H \subset \tilde{H}, \tilde{H}'$, the mentioned isomorphism sends ∇' to ∇ . \square

Let b be a regular value of f and $\{\delta_t\}_{t \in (\mathbb{C}, b)}, \delta_t \in f^{-1}(t)$ be a continuous family of cycles in the fibers of f . For an $\omega \in \tilde{H}$ the integral $\int_{\delta} \omega$ is well defined and

$$\frac{\partial}{\partial t} \int_{\delta_t} \omega = \int_{\delta_t} \nabla \omega \quad (5)$$

(see [AGV]). In fact, this formula is a bridge between topology and algebra in this paper.

Our objective in this section is to analyze the action of ∇ on H . For this purpose let us state a classical theorem. Let ω be a rational 1-form in \mathbb{C}^2 and $\bigcup_{i=1}^k \{f_i = 0\}$ be the pole locus of ω . Suppose that the multiplicity of ω along $\{f_i = 0\}$ is r_i .

Theorem 1.1 ([CM]). If ω is closed, i.e. $d\omega = 0$ then there are $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ and a polynomial g such that

1. If $r_i = 1$ then $\lambda_i \neq 0$.
2. If $r_i > 1$ then f_i does not divide g .
3. ω can be written as

$$\omega = \left(\sum_{i=1}^k \lambda_i \frac{df_i}{f_i} \right) + d \left(\frac{g}{f_1^{r_1-1} \dots f_k^{r_k-1}} \right).$$

Note that if ω has a pole of order r_∞ at the line at infinity then the degree of g is $\sum_{i=1}^k d_j(r_i - 1) + r_\infty - 1$ and $r_\infty + \sum_{i=1}^k \lambda_i d_i = 0$.

Let \mathcal{L} be the subset of \tilde{H} containing the 1-forms of the type $\sum_{i=1}^k \lambda_i \frac{df_i}{f_i}$, $\lambda_i \in \mathbb{C}$. We have the decomposition $\mathcal{L} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$, where \mathcal{L}_i contains all the terms $\lambda_j \frac{df_j}{f_j}$, $f_j = 0$ being an irreducible component of the fiber $f^{-1}(c_i)$. Note that if $f^{-1}(c_i)$ is irreducible then $\mathcal{L}_i = 0$.

Corollary 1.1. *For the pair (\tilde{H}, ∇) and a positive integer number n*

1.

$$\text{Kernel}(\nabla^n) = \left\{ \omega \in \tilde{H} \mid \omega = \sum_{j=0}^{n-1} \alpha_j t^j, \alpha_j \in \mathcal{L} \right\}.$$

2.

$$\text{Kernel}(\nabla^n) \cap H = \left\{ \omega \in H \mid \omega = \sum_{j=1}^{n-1} \sum_{i=1}^r \alpha_{ij} (t^j - c_i^j), \alpha_{ij} \in \mathcal{L}_i \right\},$$

where $\nabla^n = \nabla \circ \cdots \circ \nabla$ n -times.

Proof. Let us prove the first part by induction on n . We use the isomorphism in Lemma 1.2. If for $\omega \in \tilde{H}$, $\nabla \omega = 0$ then $d\omega = dP \wedge df$, where P is a rational function in \mathbb{C}^2 with poles along D . Now $d(\omega - Pdf) = 0$ and by Theorem 1.1 we have $\omega \in \mathcal{L}$. This proves the first part $n = 1$ of the induction.

Now if $\nabla^{n+1} \omega = 0$ then by induction $\nabla \omega = \sum_{j=0}^{n-1} \alpha_j t^j = \nabla \sum_{j=0}^{n-1} \frac{\alpha_j}{j+1} t^{j+1}$ or equivalently $\nabla(\omega - \sum_{j=0}^{n-1} \frac{\alpha_j}{j+1} t^{j+1}) = 0$. Using the case $n = 1$ we finish the proof of the first part.

Now let us prove the second part. Let $\omega = \sum_{j=1}^{n-1} \alpha_j t^j + \alpha_0 \in H$ and $\alpha_j = \sum_{i=1}^r \alpha_{ij}$ be the decomposition of α_j . We write

$$\begin{aligned} \omega &= \sum_{j=1}^{n-1} \sum_{i=1}^r \alpha_{ij} t^j + \alpha_0 \\ &= \sum_{j=1}^{n-1} \sum_{i=1}^r \alpha_{ij} (t^j - c_i^j) + \sum_{j=1}^{n-1} \sum_{i=1}^r \alpha_{ij} c_i^j + \alpha_0. \end{aligned}$$

The first summand in the above belongs to H and hence the second one belongs to $\mathcal{L} \cap H$ and so the second summand must be zero. \square

Before we go to the next section let us give three simple but important examples. The last one has a very special role in this paper. Corollary 1.1 with $n = 2$ will be used in the next section. Therefore we explain it with these examples.

Example 1.1. $f = (x^2 + y^2 - 1)x$. Since $xy = 0$, $3x^2x^i = x^i$, $y^2y^i = y^i$, $i \geq 1$ in V , $1, x, y, x^2$ form a basis for the vector space V . f has four critical points $p_1 = (0, 1), p_2 = (0, -1), p_3 = (\sqrt{1/3}, 0), p_4 = (-\sqrt{1/3}, 0)$ with three critical values $c_1 = c_2 = 0, c_3 = -2/3\sqrt{1/3}, c_4 = 2/3\sqrt{1/3}$. We define $\omega_1 = (x^2 + y^2 - 1)dx$. We have

$$\nabla\omega_1 = -\frac{xd(x^2 + y^2 - 1)}{t} = \frac{\omega_1}{t}, \quad \nabla^2\omega_1 = 0.$$

Since $f^{-1}(0)$ is the only reducible fiber of f , by Corollary 1.1(2) we know that any other 1-form in H with the property $\nabla^2\omega_1 = 0$ is some multiply of ω_1 by a constant.

Example 1.2. $f = xy(x + y - 1)$. There are four critical points $p_1 = (0, 0), p_2 = (1, 0), p_3 = (0, 1), p_4 = (1/3, 1/3)$ with two critical values $c_1 = c_2 = c_3 = 0, c_4 = -1/27$. Knowing Corollary 1.1, we can see that the 1-forms $\omega_1 = x(x + y - 1)dy, \omega_2 = y(x + y - 1)dx$ form a basis for the vector space $\{\omega \in H \mid \nabla^2\omega = 0\}$.

Example 1.3. The lines $l_p = (d - p)x + py - p(d - p) = 0$, $p = 0, 1, \dots, d$ are in a general position in \mathbb{R}^2 , i.e., they are distinct and no three of them have a common intersection point (l_p is the line through $(p, 0), (0, d - p)$). The polynomial $f = l_0l_1 \cdots l_d$ satisfies the following properties:

- (1) All the critical points of f in \mathbb{C}^2 are real and non-degenerated.
- (2) The values of f at all saddle critical points equal zero.

By a small perturbation of the lines l_i we also get the property

- (3) The values of f at center critical points are distinct.

(If two critical points associated to two polygons have the same value then try to collapse one of the polygons to a point or without volume region and conclude the above statement. See also Appendix of [Mo2] for this kind of arguments). In a real coordinates system (\tilde{x}, \tilde{y}) around a saddle (resp. center) critical point p the function f can be written as $f(p) + \tilde{x}^2 - \tilde{y}^2$ (resp. $f(p) + \tilde{x}^2 + \tilde{y}^2$). f has $a_2 = \frac{d(d+1)}{2}$ saddle critical points, $a_1 = \sum_{i=2}^d [\frac{i-1}{2}]$ center critical points with negative value and $a_3 = \frac{d(d-1)}{2} - a_1$ center critical points with positive value, where $[q]$ is the integer number satisfying $[q] - 1 < q \leq [q]$. By Corollary 1.1 the set of $\omega \in H$ with $\nabla^2\omega = 0$ is a vector space generated by

$$l_0l_1 \cdots l_{p-1}l_{p+1} \cdots l_d dl_p, \quad p = 0, 1, \dots, d - 1.$$

Note that $\sum_{p=0}^d l_0l_1 \cdots l_{p-1}l_{p+1} \cdots l_d dl_p = df = 0$ in H .

In what follows when we refer to the polynomial f in Example 1.3 we mean the one with a small perturbation of the lines l_p and hence satisfying property 3.

Remark 1.1. Brieskorn in [Br] introduced three $\mathcal{O}_{(\mathbb{C},0)}$ -modules H, H', H'' in the context of singularity of holomorphic functions $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ (The H used in this paper is the equivalent of Brieskorn's H'). After him these modules are called the Brieskorn lattices and recently the similar notions in a global context are introduced by many authors (see [Sa,DS,BD,Mo3]). In the context of differential equations H appears in the works of Petrov [Pe] on deformation of Hamiltonian equations of the type $d(y^2 + p(x))$, where p is a polynomial in x and is named by Gavrilov in [Ga] the Petrov module. For this reason I have used both names Brieskorn lattice and Petrov module for H .

Restriction of an $\omega \in H$ to each fiber defines a global section of the cohomology fiber bundle of the function f and looking in this way ∇ is the usual Gauss–Manin connection in the literature. For this reason, I have named ∇ again the Gauss–Manin connection. But of course we can name $\nabla\omega$ the Gelfand–Leray form of ω (see [AGV]).

2. Action of the monodromy

Suppose that f is a polynomial function in \mathbb{R}^2 with the properties 1–3 in Example 1.3. For a $c \in \mathbb{C}$ we define $L_c = f^{-1}(c)$ in \mathbb{C}^2 . $\{\delta_t\}_{t \in (\mathbb{C}, c)}$ with $\delta_t \in H_1(L_t, \mathbb{Z})$ denotes a continuous family of cycles.

Choose a value $b \in \mathbb{C}$ with $\text{Im}(b) > 0$ and fix a system of paths joining b with the critical values of f , subject to the condition that these paths lie in their entirety in the upper half-plane $\text{Im}(z) > 0$ except for the ends which coincide with the critical values. Now we can define a distinguished basis of vanishing cycles in $H_1(L_b, \mathbb{Z})$ (see [AGV] for the definition). Critical points of f are in one-to-one correspondence with the self-intersection points of the real curve $f = 0$, namely p_j , $a_1 + 1 \leq j \leq a_1 + a_2$, and relatively compact components of its complement, namely U_i^0 , $1 \leq i \leq a_1$, U_k^2 , $a_1 + a_2 + 1 \leq k \leq a = a_1 + a_2 + a_3$. U_i^0 contains a critical point of f with negative value and U_k^2 contains a critical point with positive value. We denote by

$$\delta_i^0, \delta_j^1, \delta_k^2, \quad 1 \leq i \leq a_1, \quad a_1 + 1 \leq j \leq a_1 + a_2, \quad a_1 + a_2 + 1 \leq k \leq a \quad (6)$$

the distinguished basis of vanishing cycles. δ_i^0, δ_k^2 are called the center vanishing cycles and δ_j^1 is called the saddle vanishing cycle.

Theorem 2.1 (S. Gusein-Zade, N. A'Campo). *After choosing a proper orientation for the cycles δ_i^σ we have*

- $\langle \delta_i^\sigma, \delta_j^\sigma \rangle = 0$;
- $\langle \delta_i^0, \delta_j^1 \rangle$ equal to the number of vertices of the polygon U_i^0 coinciding with the point p_j ;

- $\langle \delta_k^2, \delta_j^1 \rangle$ equal to the number of vertices of the polygon U_k^2 coinciding with the point p_j ;
- $\langle \delta_k^2, \delta_i^0 \rangle$ equal to the number of common edges of U_k^2 and U_i^0 .

This theorem, in an apparently local context, is proved by Gusein-Zade [Gu,Gu1] and A'Campo [AC,AC1] independently. However, the proof in our case is the same. The above theorem gives us the Dynkin diagram of f (see [AGV]).

Now let us state another theorem which we will use in this paper:

Theorem 2.2. *In the above situation, the vanishing cycles $\delta_i^0, \delta_j^1, \delta_k^2$ generate $H_1(L_b, \mathbb{Z})$ freely.*

The proof of the above theorems is classical and the reader can consult with [DN,Mo2]. Also the main core of the proof can be found in [La].

Let us compactify \mathbb{C}^2 in $\mathbb{P}^2 = \{[x; y; z] \mid (x, y, z) \in \mathbb{C}^3 - 0\}$. Here $\mathbb{C}^2 = \{[x; y; 1] \mid x, y \in \mathbb{C}\}$ and $\mathbb{L} = \{[x; y; 0] \mid x, y \in \mathbb{C}\}$ is the projective line at infinity. Let $f = f_0 + f_1 + \cdots + f_{d+1}$ be the decomposition of f to homogeneous parts. We look at f as a rational function on \mathbb{P}^2 by rewriting f as

$$f = f(x/z, y/z) = \frac{z^{d+1}f_0(x, y) + z^d f_1(x, y) + \cdots + f_{d+1}(x, y)}{z^{d+1}}.$$

The indeterminacy set of f is given by

$$\mathcal{R} = \{[x; y; 0] \mid f_{d+1}(x, y) = 0\}.$$

Now suppose that \mathcal{R} has $d+1$ distinct points. For instance, the polynomial in Example 1.3 has this property. This implies that the fibers of f intersect the line at infinity transversally. Doing just one blow-up in each point of \mathcal{R} and using Ehresmann's fibration theorem, we conclude that the map f is a C^∞ fibration on $\mathbb{C} - C$, where $C = \{c_1, \dots, c_r\}$ is the set of critical values of f . In general case we must do more blow-ups to obtain this conclusion and the set of critical points of f may be a proper subset of C . Therefore, we have the action of the monodromy on the first (co)homology group of L_b :

$$h : \pi_1(\mathbb{C} - C, b) \times H_1(L_b, \mathbb{Z}) \rightarrow H_1(L_b, \mathbb{Z})$$

Recall the system of paths in the beginning of this section. When we say the “Monodromy around a critical value” we mean the monodromy associated to the path which gets out of b , goes along γ (the path connecting b to the critical value in this system of paths), turns around the critical value counterclockwise and then comes back to b along γ . By Picard–Lefschetz formula (see [La]) the monodromy

around zero is given by

$$\delta \rightarrow \delta - \sum_{j=a_1+1}^{a_2} \langle \delta, \delta_j^1 \rangle \delta_j^1 \quad (7)$$

and the monodromy around a center critical value c_i is given by

$$\delta \rightarrow \delta - \langle \delta, \delta_i \rangle \delta_i. \quad (8)$$

For the regular value $b \in \mathbb{C} - C$, $\overline{L_b} = L_b \cup \mathcal{R}$ is a compact Riemann surface. Let I be the subgroup of $H_1(L_b, \mathbb{Z})$ generated by the cycles around the points of \mathcal{R} in $\overline{L_b}$. We have

$$I = \{ \delta \in H_1(L_b, \mathbb{Z}) \mid \langle \delta, \delta' \rangle = 0, \forall \delta' \in H_1(L_b, \mathbb{Z}) \}. \quad (9)$$

Elements of I are fixed under the action of the monodromy.

Now we are in a position such that we can look at our objects in an abstract way: We have a union of curves $\mathcal{C} = f^{-1}(0)$ in \mathbb{R}^2 . To \mathcal{C} we associate an Abelian group G ($= H_1(L_b, \mathbb{Z})$) freely generated by the symbols (6). These symbols are in one-to-one correspondence with the self-intersection points of \mathcal{C} and the relatively compact components of the complement of \mathcal{C} in \mathbb{R}^2 . We have an antisymmetric pairing $\langle \cdot \rangle$ given by the items of Theorem 2.1. Also the non-Abelian freely generated group $\pi_1(\mathbb{C} - C, b)$ acts from left on G with the rules (7),(8). We have also the subgroup I of G defined by (9). These are all we are going to need. From now on we can think about (vanishing) cycles in this abstract context.

We will apply the above arguments for the Example 1.3. For the line $l_p = 0, p = 0, 1, \dots, d$ we can associate the saddle critical points of f on $l_p = 0$ and the corresponding vanishing cycles. We rename these vanishing cycles by $\delta_j^{l_p}$, $j = 1, 2, \dots, d$ and suppose that the ordering by the index j is the same as the ordering of corresponding saddle points in the line $l_p = 0$ (there are two ways for such indexing, we choose one of them). We define

$$\delta^{l_p} = \sum_{j=1}^d (-1)^j \delta_j^{l_p} \in G, \quad p = 0, 1, \dots, d.$$

Now reindex all relatively compact polygons by $U_i, 1 \leq i \leq a_1 + a_3$. For any polygon U_i we denote by δ^i ($\in G$) the sum of vanishing cycles in the vertices of U_i . Also we denote by δ_i the vanishing cycle associated to U_i .

Lemma 2.1. *The cycles $\delta^{l_p}, 1 \leq p \leq d$ and $\delta^i, 1 \leq i \leq a_1 + a_3$ generate all saddle vanishing cycles in $G \otimes \mathbb{Q}$ and so they are linearly independent in $G \otimes \mathbb{Q}$.*

Note that the number of the cycles considered in the lemma is equal to the number of saddle vanishing cycles.

Proof. This is a nice high school problem. It would be more difficult if we assume only that the lines l_p are in a general position in \mathbb{R}^2 , i.e. no three of them have a common intersection point and no two of them are parallel. For our example we give the following hint: (1) First draw the lines for a small value of d . (2) Let $\delta_{p,p+1}$ denote the vanishing cycle associated to the intersection of l_p and l_{p+1} , $p = 0, 1, \dots, d-1$. Try to write $d \cdot \delta_{p,p+1}$ as an integral sum of δ^{lp} and δ^{lp+1} and δ^i , where i runs through the index of all polygons between (the angle less than 90°) the lines l_p and l_{p+1} . (3) Now it is easy to conclude that every vanishing cycle associated to the intersection points multiplied by d can be written as an integral sum of δ^{lp} , $0 \leq p \leq d$ and δ^i , $1 \leq i \leq a_1 + a_3$. (4) After choosing a proper sign for δ^{lp} , $0 \leq p \leq d$ prove that $\sum_{i=0}^d \delta^{lp} = 0$. \square

Now let us state the geometric meaning of δ^i and δ^{lp} .

Lemma 2.2. *We have*

1. $\delta^i = \delta_i - h_0(\delta_i)$, where h_0 is the monodromy around 0;
2. $I \otimes \mathbb{Q}$ is generated by the cycles δ^{lp} , $1 \leq p \leq d$.

Proof. The first part is a direct consequence of Picard–Lefschetz formula and Theorem 2.1.

By Theorems 2.1 and 2.2 we have $\langle \delta^{lp}, \delta' \rangle = 0$, $\forall \delta' \in H_1(L_b, \mathbb{Z})$ and so δ^{lp} is in I . By Lemma 2.1 δ^{lp} , $k = 1, 2, \dots, d$ are linearly independent in G and we know that I is freely generated of rank d . Therefore δ^{lp} , $k = 1, 2, \dots, d$ generate $I \otimes \mathbb{Q}$ freely. \square

Because of the symmetry in Example 1.3, one may conjecture that δ^{lp} is the cycle around $\{l_p = 0\} \cap \mathbb{L}$ in $\overline{L_b}$ (multiplied by a rational number). Since we do not need this statement we do not try to prove it.

Theorem 2.3. *In the Example 1.3 the action of the monodromy on a Lefschetz vanishing cycle generates the homology $H_1(\overline{L_b}, \mathbb{Q})$.*

Proof. First consider the case where δ is a center vanishing cycle. By Theorem 2.1 and Picard–Lefschetz formula the action of the monodromy generates δ_i and then δ_i 's. By Lemma 2.1 and Theorem 2.2 these cycles generate $H_1(\overline{L_b}, \mathbb{Q})$. Now suppose that δ is a saddle vanishing cycle. There is a center vanishing cycle δ' such that $\langle \delta', \delta \rangle \neq 0$. Performing a monodromy of δ around the critical value associated to δ' and subtracting the obtained cycle by δ we obtain δ' and so we fall in the first case. \square

In the beginning of this section we defined the degree of a polynomial 1-form $\omega = Pdy - Qdx$ to be the maximum of $\deg(F)$ and $\deg(Q)$. This definition is no more

useful when we look at ω as a rational 1-form in \mathbb{P}^2 or when we consider the foliation induced by ω in \mathbb{P}^2 . Let us introduce a new definition of degree. For a polynomial 1-form ω we define $\deg_1(\omega)$ to be the order of ω along the line at infinity minus two. We can see easily that if $\deg_1(\omega) \leq d$ then $\omega = Pdy - Qdx + G(xdy - ydx)$, where P, Q are two polynomials of degree at most d and G is zero or a homogeneous polynomial of degree d . Therefore $\deg_1(\omega) - \deg(\omega) = 0, 1$. Naturally for a $\omega \in H$ we define $\deg_1(\omega)$ to be the minimum of the \deg_1 's of the elements of ω . Let q be an indeterminacy point of f at the line at infinity \mathbb{L} . Recall that the fibers of f intersect \mathbb{L} transversally. Now we can choose a continuous family of cycles $\{\delta_t\}_{t \in \mathbb{C}}$ such that δ_t is a cycle in L_t around q . Latter we will need the following lemma.

Lemma 2.3. *For $\omega \in H$ the integral $\int_{\delta_t} \omega$ as a function in t is a polynomial of degree at most $\left\lfloor \frac{n}{d+1} \right\rfloor$, where $n - 2 = \deg_1(\omega)$ and $d + 1 = \deg(f)$. $\nabla^i \omega, i > \left\lfloor \frac{n}{d+1} \right\rfloor$ restricted to each fiber $f^{-1}(t)$ has not residues in \mathcal{R} and hence is a 1-form of the second type.*

Proof. We have $p(t) := \int_{\delta_t} \omega = t^{\frac{n}{d+1}} \int_{\delta_t} \frac{\omega}{f^{\frac{n}{d+1}}}$. Since the 1-form $\frac{\omega}{f^{\frac{n}{d+1}}}$ has not pole along the line at infinity, $\frac{p(t)}{t^{\frac{n}{d+1}}}$ has finite growth at $t = \infty$. Since $p(t)$ is holomorphic in \mathbb{C} , we conclude that $p(t)$ is a polynomial of degree at most $\left\lfloor \frac{n}{d+1} \right\rfloor$. The second part is a direct consequence of the first one and formula (5). \square

Let f be a polynomial and ω be a 1-form in \mathbb{C}^2 . ω is called relatively exact modulo f , or simply relatively exact if the underlying f is known, if the restriction of ω to each fiber L_b is exact. 1-forms of the type $dP + Qdf$, where P, Q are polynomials in \mathbb{C}^2 , are relatively exact. Latter we will need the following lemma:

Lemma 2.4. *If the fibers $f^{-1}(t), t \in \mathbb{C}$ of a polynomial f are topologically connected then every relatively exact 1-form ω is of the form $dP + Qdf$, where P, Q are two polynomials in \mathbb{C}^2 with $\deg(P) = \deg_1(\omega) + 2$ and $\deg(Q) = \deg_1(\omega) + 2 - \deg(f)$.*

The proof can be found in [Ga,BD,Bo]. For the assertion about the degrees see [Mo1, Theorem 4.1]. This kind of results was obtained for the first time by Ilyashenko in [II]. The main objective of this section is the following theorem.

Theorem 2.4. *In Example 1.3 let δ_t be a continuous family of vanishing cycles and ω be a degree d 1-form in \mathbb{C}^2 such that $\int_{\delta_t} \omega = 0, t \in (\mathbb{C}, b)$. Then ω is of the form*

$$\omega = l_0 \cdots l_d \alpha + d(P), \quad \alpha = \sum_{i=0}^d \lambda_i \frac{dl_i}{l_i}, \quad \lambda_i \in \mathbb{C},$$

where P is a polynomial of degree not greater than $d + 1$.

The statement of the above theorem for f can be considered as $(*)$ condition of Francoise in [Fr].

Proof. By the hypothesis and (5) we have $\int_{\delta_t} \nabla^2 \omega = 0, t \in (\mathbb{C}, b)$. Lemma 2.3 implies that $\nabla^2 \omega$ is a 1-form without residue in each fiber and Theorem 2.3 implies that $\nabla^2 \omega$ is a relatively exact 1-form and hence by Lemma 2.4 it is zero in \tilde{H} (Note that by (3) and (4) $\nabla^2 \omega$ is of the form $\eta/p(t)^2, \eta \in H$ and hence η is relatively exact). Since L_0 is the only reducible fiber of f , by Corollary 1.1 ω must be of the form $f \sum_{i=0}^d \lambda_i \frac{dl_i}{l_i} + dP + Qdf$, where P and Q are two polynomials in \mathbb{C}^2 and λ_i is a complex number. Recall that ω has degree d . By Lemma 2.4 we can suppose that P (resp. Q) has degree less than $d+2$ (resp. 1). We have $dP_{d+2} + Q_1 df_{d+1} = 0$, where P_{d+2} denotes the homogeneous part of P of degree $d+2$ and so on. If Q_1 is not identically zero then this equality implies that $f_{d+1} = Q_1^{d+1}$ which is not our case (write $d(P_{d+2} + Q_1 f_{d+1}) - f_{d+1} dQ_1 = 0$ and then conclude that P_{d+2} and f_{d+1} are polynomials in Q_1 . Since f_{d+1} is homogeneous in x and y it must be of the claimed form). Therefore, Q must be constant and P of degree less than $d+1$. We substitute $P + Qf$ by P and so we can assume that Q is zero. We obtain the desired equality. \square

3. Deformation and Melnikov functions

As a first attempt to prove Theorem 0.1 one may fix a generic $\mathcal{F} \in \mathcal{L}(d_1, \dots, d_s)$ and perform a deformation such that one of the center singularities of \mathcal{F} persists. Then one may try to find some tools for finding a logarithmic first integral for the deformed foliation. These tools in the Ilyashenko's case $\mathcal{L}(d+1)$ were Picard–Lefschetz theory of a polynomial and the classification of relatively exact 1-forms modulo the polynomial. Developing these tools for a generic point of other irreducible components of $\mathcal{M}(d)$, for instance $\mathcal{L}(d_1, \dots, d_s)$, seems to be difficult and wasting the time.

In this section we want to explain this idea that it is not necessary to take a generic point of $\mathcal{L}(d_1, \dots, d_s)$. For instance, suppose that the variety $\mathcal{L}(d_1, \dots, d_s)$ has a point $\mathcal{F}(df)$ which is Hamiltonian. This point may lie in other irreducible components of $\mathcal{M}(d)$. Now the idea is to deform $\mathcal{F}(df)$ in such a way that one of its centers persists and since we can develop our tools for the Hamiltonian system $\mathcal{F}(df)$, we can calculate the tangent cone of $\mathcal{M}(d)$ in $\mathcal{F}(df)$. Now if the tangent cone of one of the branches of $\mathcal{L}(d_1, \dots, d_s)$ at $\mathcal{F}(df)$ is an irreducible component of the tangent cone of $\mathcal{M}(d)$ at $\mathcal{F}(df)$, then $\mathcal{L}(d_1, \dots, d_s)$ is irreducible component of $\mathcal{M}(d)$ locally and hence globally.

Now we are going to realize this idea for the Hamiltonian foliation with the polynomial in Example 1.3. This will lead to the proof of our main theorem. In this section \mathcal{L} denotes the set of rational 1-forms of the type $\sum_{i=0}^d \lambda_i \frac{dl_i}{l_i}$ and \mathcal{P}_n (resp. \mathcal{P}_*) denotes the set of polynomials of degree not greater than n (resp. arbitrary degree) in \mathbb{C}^2 .

Let f be the polynomial considered in Example 1.3 and

$$\mathcal{F}_\varepsilon : \omega_\varepsilon = df + \varepsilon^k \omega_k + \varepsilon^{k+1} \omega_{k+1} + \cdots + \varepsilon^{2k} \omega_{2k} + h.o.t. = 0, \quad \omega_k \neq 0, \quad (10)$$

where $\omega_i, i = k, k+1, \dots$ are polynomial 1-forms in \mathbb{C}^2 and k is a natural number, be a one-parameter degree d deformation of $\mathcal{F}(df)$.

Definition 3.1. Let p be a center singularity of $\mathcal{F} = \mathcal{F}(df)$. p is called a persistent center if there exists a sequence $\varepsilon_i, i = 1, 2, 3, \dots, \varepsilon_i \rightarrow 0$ such that the singularity of $\mathcal{F}_{\varepsilon_i}$ near $p_0 = p$, namely p_{ε_i} , is center for all i .

Later we will see that if p is persistent then p_ε is center for all ε . Let $I = \{0, 1, \dots, d\}$. For an equivalence relation J in I we denote by J_1, J_2, \dots, J_{s_J} all equivalence classes of J and we define $f_i^J = \prod_{j \in J_i} l_j, i = 1, 2, \dots, s_J$. Our main theorem in this section is the following:

Theorem 3.1. If p is a persistent center in the degree d deformation (10) then ω_k is of the form

$$\omega_k = l_0 l_1 \cdots l_d \sum_J \sum_{i=1}^{s_J} \left(\lambda_i^J \frac{df_i^J}{f_i^J} + \frac{A_i^J}{f_i^J} \right),$$

where in the first sum J runs through all equivalence relations in I , for each J the complex numbers $\lambda_i^J, i = 1, 2, \dots, s_J$ are distinct and $A_i^J \in \mathcal{P}_{\deg(f_i^J)}$.

Let $\delta_t, t \in (\mathbb{C}, b)$ be a continuous family of vanishing cycles around p and Σ be a transverse section to \mathcal{F} at some point of δ_b . We write the Taylor expansion of the deformed holonomy $h_\varepsilon(t)$

$$h_\varepsilon(t) - t = M_1(t)\varepsilon + M_2(t)\varepsilon^2 + \cdots + M_i(t)\varepsilon^i + h.o.t.$$

$M_i(t)$ is called the i th Melnikov function of the deformation. If the center p is persistent under the deformation then $M_i = 0$ for all i . But we do not need to use all these equalities. For instance in Ilyashenko's case $\mathcal{L}(d+1), k=1$ we need only $M_1 = 0$. To prove Theorem 3.1 and our main theorem we will need to use $M_k = M_{k+1} = \cdots = M_{2k} = 0$.

Lemma 3.1. Let M_i be the i th Melnikov function associated to deformation (10). We have: (1) $M_1 = M_2 = \cdots = M_{k-1} = 0$. (2) If Σ is parameterized by the image of f , i.e. $t = f(z), z \in \Sigma$ then

$$M_k(t) = - \int_{\delta_t} \omega_k.$$

(3) If $M_k = M_{k+1} = \cdots = M_{2k-1} = 0$ then

$$\omega_i = f \alpha_i + dP_i, \quad P_i \in \mathcal{P}_{d+1}, \quad \alpha_i \in \mathcal{L}, \quad k \leq i \leq 2k-1.$$

(4) Moreover, if the transverse section is parameterized by the image of a branch of $\ln f$, i.e. $t = \ln f(z)$, $z \in \Sigma$ then

$$M_{2k}(t) = - \int_{\delta_t} \frac{\omega_{2k}}{f} - \frac{P_k}{f} \alpha_k.$$

This theorem can be considered as a generalization of Francoise recursion formula (see [Fr]). Note that for our polynomial, which is a product of lines, we have Theorem 2.4 instead of Francoise (*) condition.

Proof. Let $\delta_{t, h_\varepsilon(t)}$ be the path connecting t and $h_\varepsilon(t)$ along δ_b in the leaf of \mathcal{F}_ε through t . We take the integral $\int_{\delta_{t, h_\varepsilon(t)}}$ of (10). Now the equalities associated to the coefficients of ε^i , $1 \leq i \leq k-1$ prove the first part. The equality associated to the coefficient of ε^k proves the second part (for more details see [Ro, Mo]).

We prove the third and fourth part by induction on i . First $i = k$. $M_k = 0$ implies that $\int_{\delta_t} \omega_k = 0$, $t \in (\mathbb{C}, b)$ and so by Theorem 2.4 ω_k is of the form

$$\omega_k = f \alpha_k + dP_k, \quad \alpha_k \in \mathcal{L}, \quad P_k \in \mathcal{P}_{d+1}. \quad (11)$$

Now let us suppose that

$$\omega_j = f \alpha_j + dP_j, \quad \alpha_j = \sum_{p=0}^d \lambda_{j,p} \frac{dl_p}{l_p}, \quad k \leq j \leq i.$$

Let $\bar{\omega}_\varepsilon = \omega_\varepsilon/f$, $\bar{\omega} = \omega/f$ and so on. With this new notation we have

$$\tilde{\omega}_j = \alpha_j + \overline{dP_j} = d(\bar{P}_j + \ln l_1^{j,1} \dots l_{d+1}^{j,d+1}) + \bar{P}_j \overline{df} = dg_j + \bar{P}_j \overline{df}.$$

From now on suppose that Σ is parameterized by the image of $\ln f$. We have

$$\begin{aligned} & (1 - \bar{P}_k \varepsilon^k) \dots (1 - \bar{P}_i \varepsilon^i) \bar{\omega}_\varepsilon, \\ & (1 - \bar{P}_k \varepsilon^k) \dots (1 - \bar{P}_i \varepsilon^i) (\overline{df} + \varepsilon^k \bar{\omega}_k + \varepsilon^{k+1} \bar{\omega}_{k+1} + \dots + \varepsilon^{2k} \bar{\omega}_{2k} + h.o.t.) \\ & = \overline{df} + \varepsilon^k dg_k + \varepsilon^{k+1} dg_{k+1} + \dots + \varepsilon^i dg_i \\ & \quad + \varepsilon^{i+1} \bar{\omega}_{i+1} + \dots + \varepsilon^{2k-1} \bar{\omega}_{2k-1} + \varepsilon^{2k} (\bar{\omega}_{2k} - \bar{P}_k \bar{\omega}_k) + h.o.t. \end{aligned} \quad (12)$$

We take the integral $\int_{\delta_{t, h_\varepsilon(t)}}$ of (12). Now the equality associated to the coefficients of ε^{i+1} is $M_{i+1}(t) + \int_{\delta_t} \bar{\omega}_{i+1} = 0$. $M_{i+1} = 0$ and Theorem 2.4 imply that ω_{i+1} is of the desired form. In the last step $i = 2k-1$ the fourth part of the lemma is proved. Note that $\int_{\delta_t} \bar{\omega}_{2k} - \bar{P}_k \bar{\omega}_k = \int_{\delta_t} \bar{\omega}_{2k} - \bar{P}_k \alpha_k$. \square

Now $M_{2k} = 0$ implies that $\int_{\delta_t} f \omega_{2k} - P_k f \alpha_k = 0$. $\omega_{2k} - P_k f \alpha_k$ has a pole of order at most $2d+3$ at the line at infinity. Therefore by Lemma 2.3 $\nabla^3(\omega_{2k} - P_k f \alpha_k)$

restricted to the fibers $\overline{f^{-1}(b)}$ has no residues. By Theorem 2.3 and Corollary 1.1 we conclude that

$$f\omega_{2k} - P_k f\alpha_k = \alpha_1 f + \alpha_2 f^2 + dg + pdf, \quad g, p \in \mathcal{P}_*, \quad \alpha_i \in \mathcal{L}, \quad i = 1, 2.$$

The restriction of the above equality to the $L_0 = f^{-1}(0)$ implies that g is constant on L_0 . Since L_0 is connected in \mathbb{C}^2 , we conclude that $dg = d(fg'), g' \in \mathcal{P}_*$. From now on we write $\lambda_{k,i} = \lambda_i$. The above equality modulo multiplications by l_i gives us

$$l_i \mid \lambda_i P_k + g', \quad i = 0, 1, \dots, d. \quad (13)$$

Let $I = \{0, 1, \dots, d\}$. Define $i \cong j$ if $\lambda_i = \lambda_j$. \cong is an equivalence relation. Let J_1, J_2, \dots, J_s be the equivalence classes of \cong . We define $f_i = \prod_{j \in J_i} l_j$. Note that we have $f = f_1 f_2 \cdots f_s$. The following lemma finishes the proof of Theorem 3.1.

Lemma 3.2. *In the above situation, P_k must be of the form*

$$P_k = f \sum_{i=1}^s \frac{A_i}{f_i}, \quad A_i \in \mathcal{P}_{\#J_i}. \quad (14)$$

Proof. Let $d_i = \deg(f_i)$. Eq. (13) implies that P_k is zero in $\{f_i = 0\} \cap \{f_j = 0\}$. The space of $P \in \mathcal{P}_{d+1}$ vanishing in $\{f_i = 0\} \cap \{f_j = 0\}$ for all $1 \leq i < j \leq s$, namely G , is of dimension $\frac{(d+2)(d+3)}{2} - \sum_{1 \leq i < j \leq s} d_i d_j$. (The matrix $[P_m(B_n)]$ where P_m runs in \mathcal{P}_{d-1} and B_m in the intersection points of the lines l_i has non-zero determinant, otherwise there would be a polynomial P of degree not greater than $d-1$ vanishing in all B_m which is a contradiction, because $P=0$ intersects a line at most in $d-1$ points. Therefore, the map $\psi : \mathcal{P}_{d+1} \rightarrow \mathbb{C}^{d(d+1)/2}$, $\psi(P) = (P(B_m))$ is surjective and hence the map $\psi' : \mathcal{P}_{d+1} \rightarrow \mathbb{C}^{\sum_{1 \leq i < j \leq s} d_i d_j}$, $\psi'(P) = (P(B_m))$ is surjective, where in the second map B_m runs in the intersection points of f_i 's.) But the space of polynomials in (14) is a subset of G and has dimension $-1 + \sum_{i=1}^s \frac{(d_i+1)(d_i+2)}{2}$. Since $d+1 = \sum_{i=1}^s d_i$, these two numbers are equal and so P_k must be of form (14). \square

4. Proof of Theorem 0.1

Let $(X, 0)$ be a germ of an analytic variety in $(\mathbb{C}^n, 0)$. The tangent cone $TC_0 X$ of X at 0 is defined as follows: Let $\gamma : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$ be an analytic map such that its image lies in X and has the Taylor series $\gamma = \omega \varepsilon^l + \omega' \varepsilon^{l+1} + \cdots$, $\omega, \omega', \dots \in \mathbb{C}^n$. T_l is the set of all ω and $TC_0 X = \bigcup_{l=1}^{\infty} T_l$.

We have $\mathbb{C}.TC_0 X = TC_0 X$ therefore we can projectivize $TC_0 X$ and obtain a subset, namely Y , of \mathbb{P}^{n-1} . Suppose that X is irreducible. Let $\pi : M \rightarrow (\mathbb{C}^n, 0)$ be a blow-up at 0 with the divisor $\pi^{-1}(0) \cong \mathbb{P}^{n-1}$. The closure \tilde{X} of $\pi^{-1}(X - \{0\})$ in M is

an irreducible analytic set and we can see easily that Y is isomorphic to the intersection of \tilde{X} and $\mathbb{P}^{n-1} \subset M$, and so it is algebraic compact subset of \mathbb{P}^{n-1} . Moreover, since $\dim(Y) \geq \dim(\tilde{X}) + \dim \mathbb{P}^{n-1} - n$ (see [Ke, Theorem 3.6.1]) and Y cannot be the whole \mathbb{P}^{n-1} , Y is of pure dimension $\dim X - 1$, i.e. each irreducible component of Y is of dimension $\dim X - 1$. We conclude that TC_0X is an algebraic subset of \mathbb{C}^n of pure dimension $\dim(X)$. If X is smooth then TC_0X is the usual tangent space of X at 0 and hence it is a vector space. For more information about the tangent cone of a singularity and its definition by the leading terms of the polynomial defining the singularity see [Mu, Ke, Section 6.2].

The variety $\mathcal{L}(d_1, \dots, d_s)$ is parameterized by

$$\begin{aligned} \tau : \mathbb{C}^s \times \mathcal{P}_{d_1} \times \dots \times \mathcal{P}_{d_s} &\rightarrow \mathcal{F}(d), \\ \tau(\lambda_1, \dots, \lambda_s, f_1, \dots, f_s) &= f_1 \cdots f_s \sum_{i=1}^s \lambda_i \frac{df_i}{f_i} \end{aligned} \quad (15)$$

and so it is irreducible. Let J be an equivalence relation in $I = \{0, 1, \dots, d\}$ with s equivalence classes, namely J_1, \dots, J_s . Let also f be the polynomial in Example 1.3 and $\mathcal{F}_0 = \mathcal{F}(df)$. In a neighborhood of \mathcal{F}_0 in $\mathcal{F}(d)$, $\mathcal{L}(d_1, \dots, d_s)$ has many irreducible components corresponding to the J 's as follows:

The above parameterization near $(1, \dots, 1, \prod_{i \in J_1} l_i, \dots, \prod_{i \in J_s} l_i)$ determines an irreducible component, namely $\mathcal{L}(d_1, \dots, d_s)_J$, of $(\mathcal{L}(d_1, \dots, d_s), \mathcal{F}_0)$ corresponding to J . Fix one of these branches and name it X . Now to prove our main theorem it is enough to prove that X is an irreducible component of $(\mathcal{M}(d), \mathcal{F}_0)$.

What we have proved in Theorem 3.1 is

$$TC_{\mathcal{F}_0} \mathcal{M}(d) = \bigcup TC_{\mathcal{F}_0} \mathcal{L}(d_1, \dots, d_s)_J,$$

where the union is taken over all $1 \leq d_i \leq d+1$, $1 \leq i \leq s \leq d+1$, $\sum_{i=1}^s d_i = d+1$ and all equivalence relations J . Now let $X \subset X'$, where X' is an irreducible component of $(\mathcal{M}(d), \mathcal{F}_0)$. Since the above union is the decomposition of $TC_{\mathcal{F}_0} \mathcal{M}(d)$ to irreducible components, the irreducible component of $TC_{\mathcal{F}_0} X'$ containing $TC_{\mathcal{F}_0} X$ must be a subset of $TC_{\mathcal{F}_0} X$ and so is equal $TC_{\mathcal{F}_0} X$. An irreducible component of $TC_{\mathcal{F}_0} X'$ is of dimension $\dim(X)$ and so X' is of dimension $\dim(X)$. Since $X \subset X'$ and X, X' are irreducible, we conclude that $X = X'$. \square

Limit cycles and Bautin ideals: Let $\mathcal{F}_0 \in \mathcal{M}(d)$, p_0 be a center singularity of \mathcal{F}_0 , $\delta_t, t \in (\mathbb{C}, 0)$ be a continuous family of cycles invariant by \mathcal{F}_0 around p_0 and Σ be a transverse section to \mathcal{F}_0 at some point of δ_0 . Let also (\mathbb{C}^μ, ψ) be an affine chart of $\mathcal{F}(d)$ with $\psi(\mathcal{F}_0) = 0$. We use also ψ for the points in \mathbb{C}^μ . For instance we denote by \mathcal{F}_ψ the foliation associated to $\psi \in \mathbb{C}^\mu$ by this affine chart.

The holonomy of \mathcal{F}_0 along δ_0 in Σ is identity. Now considering a ψ near 0, we have the holonomy h_ψ of \mathcal{F}_ψ along δ_0 in Σ which is called the deformed holonomy. We write the Taylor expansion

$$h_\psi(t) - t = \sum_{i=0}^{\infty} a_i(\psi) t^i.$$

The ideal generated by $a_i(\psi)$, $0 \leq i$ is called the Bautin ideal of δ_0 in the deformation space $\mathcal{F}(d)$. If $\psi \in \text{Zero}(I)$ then the holonomy of \mathcal{F}_ψ along δ_0 is identity. Using Hartogs extension theorem (see also [III]), one can see that the singularity p_ψ near p_0 is center and so $\mathcal{F}_\psi \in \mathcal{M}(d)$. We conclude that $\text{zero}(I) \subset \mathcal{M}(d)$.

The center p_0 of \mathcal{F}_0 is called stable if for any deformation \mathcal{F}_τ , $\tau \in (\mathbb{C}^k, 0)$ of \mathcal{F}_0 inside $\mathcal{M}(d)$, the deformed singularity p_τ is again a center. Let $\mathcal{F}_0 \in \mathcal{M}(d)$ and \mathcal{F}_τ be a deformation of \mathcal{F}_0 inside $\mathcal{M}(d)$. Since each \mathcal{F}_τ has at least one center, there is a sequence p_{τ_i} of centers converging to a singularity of \mathcal{F}_0 . We conclude that the deformed holonomy along the vanishing cycles around p_0 is identity and p_τ is center for all τ . From this argument we conclude that every \mathcal{F}_0 with $(\mathcal{M}(d), \mathcal{F}_0)$ irreducible has at least one stable center. In particular, generic points of irreducible components have stable centers. It is an interesting problem to show that a generic point of $\mathcal{L}(d_1, \dots, d_s)$ has $d^2 - \sum_{i < j} d_i d_j$ stable centers. For the stable center p_0 we have $\text{zero}(I) = (\mathcal{M}(d), \mathcal{F}_0)$, where I is the Bautin ideal associated to a vanishing cycle around p_0 and the deformation space $\mathcal{F}(d)$.

Now let X be an irreducible component of $\mathcal{M}(d)$, $\mathcal{F} \in X - \text{sing}(\mathcal{M}(d))$ be a real foliation, i.e. its equation has real coefficients, p be a real center singularity and δ_t , $t \in (\mathbb{R}, 0)$ be a family of real vanishing cycles around p . The cyclicity of δ_0 in a deformation of \mathcal{F} inside $\mathcal{F}(d)$ is greater than $\text{codim}_{\mathcal{F}(d)}(X) - 1$. Roughly speaking, the cyclicity of δ_0 is the maximum number of limit cycles appearing near δ_0 after a deformation of \mathcal{F} in $\mathcal{F}(d)$. The proof of this fact and the exact definition of cyclicity can be found in [Ro]. $\text{codim}_{\mathcal{F}(d)}(\mathcal{L}(d+1)) - 1 = \frac{(d+2)(d-1)}{2}$ and this is the number obtained by Ilyashenko in [II]. Now let $X = \mathcal{L}(d_1, \dots, d_s)$ and $\mathcal{F} = \mathcal{F}(f \sum_{i=1}^s \lambda_i \frac{df_i}{f_i}) \in \mathcal{L}(d_1, \dots, d_s) - \text{sing}(\mathcal{M}(d))$. Suppose that λ_i 's and the coefficients of f_i 's are real numbers and \mathcal{F} has a (real) center singularity at $0 \in \mathbb{R}^2$. We conclude that

Corollary 4.1. *Suppose that $s > 1$. The cyclicity of δ_0 in a deformation of \mathcal{F} in $\mathcal{F}(d)$ is not less than*

$$(d+1)(d+2) - \sum_{i=1}^s \left(\frac{(d_i+1)(d_i+2)}{2} \right) - 1.$$

Note that the above lower bound reaches to its maximum when $d_1 = d_2 = \dots = d_s = 1$, $s = d+1$. In this case the cyclicity of δ_0 is not less than $d^2 - 2$. Until the time of writing this paper, the best upper bound for the cyclicity of a vanishing cycle of a Hamiltonian equation is the Mardesic's result $\frac{d^4+d^2-2}{2}$ in [Ma]. Results for the cyclicity of period annulus are obtained by many authors, the most complete concerns the Hamiltonian case with $d = 2$ (see [Ga2] and references given there).

We can state center conditions for an arbitrary algebraically closed field k instead of \mathbb{C} . The notations in the introduction can be developed for k except the center singularity. Suppose that the origin $0 = (0, 0) \in k^2$ is a reduced singularity of

$\mathcal{F}(\omega) \in \mathcal{F}(d)$. It is called a center singularity of ω if there is a formal power series $f = xy + f_3 + f_4 + \cdots + f_n + \cdots$, where f_n is a homogeneous polynomial of degree n and with coefficients in k , such that $df \wedge \omega = 0$. A singularity p of ω is called center if the origin is a center singularity of $i^* \omega$, where i is the linear transformation $a \rightarrow a + p$ in k^2 . Our definition is complete. Now let $k = \mathbb{C}$ and the origin is a center singularity of a 1-form ω . By Theorem A in [MaMo] the existence of the formal series f implies the existence of a convergent one, namely g . Using the complex Morse theorem we find a coordinates system (\tilde{x}, \tilde{y}) around the origin such that $g = \tilde{x}^2 + \tilde{y}^2$. So our definition of a center singularity coincides with the definition in the introduction. Now the proof of the fact that $\mathcal{M}(d)$ is an algebraic subset of $\mathcal{F}(d)$ is a slight modification of the proof in [Mol].

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